REARRANGEMENTS OF SERIES OF FUNCTIONS

BY
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1. **Introduction and the statement of the main result.** It is well known [3, p. 301] that a conditionally convergent series of real numbers can be rearranged in such a way so as to converge to any preassigned value. Suppose now we have a series of functions

$$\sum_{n=1}^{\infty} x_n(t), \qquad 0 \leqq t \leqq 1. \tag{1}$$

In this paper we shall be concerned with the studies of rearrangements of such series, the convergence being that of $L_2(0, 1)$. The work is motivated by the paper of E. Steinitz [4], who considered the rearrangements of conditionally convergent series of vectors in the finite dimensional Euclidean spaces. We are going to prove the following result:

THEOREM 1. *Suppose $x_n(t)$ is a sequence of real valued functions, belonging to the real space $L_2(0, 1)$. Suppose also that*:

(a) *the series* (1) *converges in norm to some $x \in L_2$*,

(b) $\sum \|x_n\| = +\infty$,

(c) $\sum \|x_n\|^2 < \infty$,

(d) *the linear subspace $M = \{y \in L_2 : \sum |(x_n, y)| < \infty\}$ is closed ((a, b) is the real inner product of a and b).*

Then there exists a closed linear subspace N and a function $x_0 \in L_2$ such that:

I. *any rearrangement of* (1), *which converges in norm, must have the limit of the form $x_0 + z$, where $z \in N$*;

II. *for any $z \in N$, there exists a rearrangement of* (1), *which converges in norm to $x_0 + z$.*

In fact $N = M^{\perp}$ i.e., $N \oplus M = L_2$.

2. **Proof of Theorem 1.** First we need several lemmas.

LEMMA 1. *Let $x_1, \ldots, x_{n+1}$ be $n+1$ linearly dependent elements in L_2 and let*

$$x = \alpha_1 x_1 + \cdots + \alpha_{n+1} x_{n+1}, \qquad 0 \leqq \alpha_i \leqq 1.$$

Then we can express x as

$$x = \gamma_1 x_1 + \cdots + \gamma_{n+1} x_{n+1}, \qquad 0 \leqq \gamma_i \leqq 1$$

and at least one $\gamma_i = 0$ or 1.

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Proof. This Lemma is proved in [4, p. 167] for the case when $x_i \in R^n$ and the proof carries verbatim to the present situation.

LEMMA 2. *Let* $x_1, x_2, \ldots, x_n \in L_2$ *and let*

$$x = \lambda_1 x_1 + \cdots + \lambda_n x_n, \qquad 0 \leqq \lambda_1 \leqq 1.$$

Then there exists a vector $x' \in L_2$ *of the form*

$$x' = \delta_1 x_1 + \delta_2 x_2 + \cdots + \delta_n x_n, \qquad \delta_i = 0 \quad or \quad 1$$

such that $\|x - x'\|^2 \leqq \|x_1\|^2 + \cdots + \|x_n\|^2$.

Proof. We proceed by induction on n. The case $n=1$ is clear since $|\lambda_1| \leqq 1$. Suppose then the lemma is true for any n vectors in L_2 and let

$$x = \lambda_1 x_1 + \cdots + \lambda_n x_n + \lambda_{n+1} x_{n+1}, \qquad 0 \leqq \lambda_i \leqq 1. \tag{2}$$

We may write $x_{n+1} = y_{n+1} + z_{n+1}$, where $y_{n+1} \in \operatorname{sp}\{x_1, \ldots, x_n\}$ and $(z_{n+1}, x_i) = 0$, $i = 1, 2, \ldots, n$. Clearly

$$\|x_{n+1}\|^2 = \|y_{n+1}\|^2 + \|z_{n+1}\|^2, \tag{3}$$

$$x - \lambda_{n+1} z_{n+1} = \lambda_1 x_1 + \cdots + \lambda_n x_n + \lambda_{n+1} y_{n+1}, \qquad 0 \leqq \lambda_i \leqq 1 \tag{4}$$

and the vectors $x_1, \ldots, x_n, y_{n+1}$ are linearly dependent. Using Lemma 1 we can write (4) as

$$x - \lambda_{n+1} z_{n+1} = \gamma_1 x_1 + \cdots + \gamma_n x_n + \gamma_{n+1} y_{n+1}, \qquad 0 \leqq \gamma_i \leqq 1 \tag{5}$$

and at least one $\gamma_{i_0} = 0$ or 1.

We divide the remaining proof into several cases.

Case 1. $i_0 = n+1$, $\gamma_{n+1} = 0$. By the inductive hypothesis there exists a vector $x' = \delta_1 x_1 + \cdots + \delta_n x_n$, $\delta_i = 0$ or 1, such that

$$\|x - \lambda_{n+1} z_{n+1} - x'\|^2 \leqq \|x_1\|^2 + \cdots + \|x_n\|^2. \tag{6}$$

Since $x = (x - \lambda_{n+1} z_{n+1}) + \lambda_{n+1} z_{n+1}$ and z_{n+1} is orthogonal to $x - \lambda_{n+1} z_{n+1}$ (equation 5 and the conditions of this case) and x', we get from (6) and (3)

$$\|x - x'\|^2 = \|x - \lambda_{n+1} z_{n+1} - x'\|^2 + |\lambda_{n+1}|^2 \|z_{n+1}\|^2 \leqq \|x_1\|^2 + \cdots + \|x_{n+1}\|^2.$$

Case 2. $i_0 < n+1$ and $\gamma_{i_0} = 0$. We may assume without the loss of generality that $i_0 = 1$. By the inductive hypothesis there is a vector $x'' = \delta_2 x_2 + \cdots + \delta_{n+1} y_{n+1}$, $\delta_i = 0$ or 1, such that

$$\|x - \lambda_{n+1} z_{n+1} - x''\| \leqq \|x_2\|^2 + \cdots + \|x_n\|^2 + \|y_{n+1}\|^2. \tag{7}$$

Let $x' = x'' + \delta_{n+1} z_{n+1} = \delta_2 x_2 + \cdots + \delta_{n+1} x_{n+1}$. Since z_{n+1} is orthogonal to $x_1, x_2, \ldots, x_n, y_{n+1}$, and $|\lambda_{n+1} - \delta_{n+1}| \leqq 1$ we obtain from (7) and (3)

$$\begin{aligned} \|x - x'\|^2 &= \|x - \lambda_{n+1} z_{n+1} - x'' + (\lambda_{n+1} - \delta_{n+1}) z_{n+1}\|^2 \\ &= \|x - \lambda_{n+1} z_{n+1} - x''\|^2 + |\lambda_{n+1} - \delta_{n+1}|^2 \|z_n\|^2 \\ &\leqq \|x_1\|^2 + \|x_2\|^2 + \cdots + \|x_n\|^2 + \|y_{n+1}\|^2 + \|z_{n+1}\|^2. \end{aligned}$$

Since the last 2 terms add up to $\|x_{n+1}\|^2$ we get the result.

Case 3. $i_0=n+1$, $\gamma_{n+1}=1$. Here

$$x-\lambda_{n+1}z_{n+1} = \gamma_1 x_1+\cdots+\gamma_{n+1}x_n+y_{n+1}. \tag{8}$$

Let $x''=\delta_1 x_1+\cdots+\delta_n x_n$, $\delta_i=0$ or 1, be such that

$$\|x-\lambda_{n+1}z_{n+1}-y_n-x''\|^2 \leqq \|x_1\|^2+\cdots+\|x_n\|^2 \tag{9}$$

and put $x'=x''+x_{n+1}=x_{n+1}+y_{n+1}+z_{n+1}$. Since $|\lambda_{n+1}-1|\leqq 1$ and $z_i\perp x_i,\ldots, x_n, y_{n+1}$, we get from (9) and (8)

$$\begin{aligned}\|x-x'\|^2 &= \|x-\lambda_{n+1}z_{n+1}-y_{n+1}-x''+(\lambda_{n+1}-1)z_{n+1}\|^2\\ &= \|x-\lambda_{n+1}z_{n+1}-y_{n+1}-x''\|^2+|\lambda_{n+1}-1|^2\|z_{n+1}\|^2\\ &\leqq \|x_1\|^2+\cdots+\|x_{n+1}\|^2.\end{aligned}$$

Case 4. $i<n+1$. Here again we assume that $i_0=1$ and consequently $\gamma_1=1$. We write now

$$x-\lambda_{n+1}z_{n+1}-x_1 = \gamma_2 x_2+\cdots+\gamma_n x_n+\gamma_{n+1}y_{n+1}.$$

Let $x''=\delta_2 x_2+\cdots+\delta_n x_n+\delta_{n+1}y_{n+1}$, $\delta_i=0$ or 1, be such that

$$\|x-\lambda_{n+1}z_{n+1}-x_1-x''\| \leqq \|x_2\|^2+\cdots+\|y_{n+1}\|^2. \tag{10}$$

Put

$$x' = x_1+x''+\delta_{n+1}z_{n+1} = x_1+\delta_2 x_2+\cdots+\delta_{n+1}x_{n+1}.$$

Noting again that $z_{n+1} \perp x_1,\ldots, y_{n+1}$, $|\lambda_{n+1}-\delta_{n+1}|\leqq 1$ we get

$$\begin{aligned}\|x-x'\|^2 &= \|x-\lambda_{n+1}z_{n+1}-x_1-x''+(\lambda_{n+1}-\delta_{n+1})z_{n+1}\|^2\\ &\leqq \|x_2\|^2+\cdots+\|y_{n+1}\|^2+|\lambda_{n+1}-\delta_{n+1}|^2\|z_{n+1}\|^2 \leqq \|x_1\|^2+\cdots+\|x_{n+1}\|^2.\end{aligned}$$

This proves Lemma 2.

LEMMA 3. *Let $X=\{x_n\}$ be a sequence of elements of L_2. Let*
$P(X)=\{x_{i_1}+x_{i_2}+\cdots+x_{i_k} : i_1<i_2<\cdots<i_k\}$,
$Q(X)=\text{co}\,P(X)$ *(convex hull of $P(X)$).*
$R(X)=\{\gamma_1 x_{i_1}+\cdots+\gamma_k x_{i_k} : 0\leqq\gamma_i\leqq 1,\ i_1<i_2<\cdots<i_k\}$.
Then $Q(X)\subset R(X)$.

Proof. It is enough to show that $R(X)$ is convex, since $R(X)\supset P(X)$. Let now $y, z \in R(X)$. We may assume

$$y = \gamma_1 x_{i_1}+\gamma_2 x_{i_2}+\cdots+\gamma_k x_{i_k}, \qquad z = \delta_1 x_{i_1}+\delta_2 x_{i_2}+\cdots+\delta_k x_{i_k}$$

by inserting the terms $0\cdot x_j$ if necessary.

Let now $0\leqq\lambda\leqq 1$. We have

$$\lambda y+(1-\lambda z) = \sum_j [\lambda\gamma_j+(1-\lambda)\delta_j]x_{i_j}$$

and since $0\leqq\lambda\gamma_j+(1-\lambda)\delta_j\leqq\lambda+(1-\lambda)=1$ we get the result.

LEMMA 4. *Let N be a closed linear subspace of L_2 and let B be a convex subset of N. Suppose that for any $x \in N$ and any $T>0$, there exist elements b_1 and b_2 in B so that $(x, b_1) \leqq -T$ and $(x, b_2) \geqq T$. Then B is dense in N. (We recall that our L_2 is a real space.)*

Proof. Consider N as a Hilbert space and suppose the closure of B is different from N. Let $K=\{x \in N : \|x-x_0\| < r\} \subset N \setminus \bar{B}$, for some $x_0 \in N$ and $r>0$. Since K and $\bar{B}$ are convex and K has an interior point (in the relative topology of N), there exists a continuous linear functional x' such that $x'(x) \leqq c \leqq x'(y)$, for all $x \in \bar{B}$, $y \in K$ for some constant c. (See [2, p. 412].) By the Riesz representation theorem for the Hilbert spaces, $x'(x)=(x, x')$ for some $x' \in N$. But this implies that $(x, x') \leqq c$ for all $x \in B$, which contradicts the hypothesis of the theorem.

LEMMA 5. *Suppose $x_i \in L_2$, $\|x_i\| \leqq M$, $i=1, 2, \ldots, n$. Let*

$$x_1+x_2+\cdots+x_n = a. \tag{11}$$

Then we can rearrange the order of x_i's, say into $\{x_1', x_2', \ldots, x_n'\}$, such that

$$\|x_1'+x_2'+\cdots+x_p'\|^2 \leqq \|x_1\|^2+\cdots+\|x_n\|^2+\|a\|(\|a\|+2M), p = 1, 2, \ldots, n. \tag{12}$$

Proof. First assume that $a=0$ and call the right hand side of (12) K. We proceed to rearrange $x_1, \ldots, x_n$ as follows. Let $x_1'=x_1$. Clearly $\|x_1'\|^2 \leqq K$. On account of (11) we have $\sum (x_1', x_i)=(x_1', 0)=0$ and the first term of the sum is equal to $\|x_1'\|^2 \geqq 0$. Hence for some x_2' among $x_2, \ldots, x_n$ we must have $(x_1', x_2') \leqq 0$. From this it follows that

$$\|x_1'+x_2'\|^2 = \|x_1'\|^2+2(x_1', x_2')+\|x_2'\|^2 \leqq \|x_1'\|^2+\|x_2'\|^2 \leqq K.$$

Consider next $\sum (x_1'+x_2', x_i)=(x_1'+x_2', 0)=0$. The first 2 terms add up to $\|x_1'+x_2'\|^2 \geqq 0$; hence for some x_3' among x's different from x_1', x_2' we must have $(x_1'+x_2', x_3') \leqq 0$. So

$$\|x_1'+x_2'+x_3'\|^2 = \|x_1'+x_2'\|^2+2(x_1'+x_2', x_3')+\|x_3'\|^2 \leqq K.$$

Continuing in this fashion we get the result.

Suppose now $a \neq 0$. Then $\sum (x_i-n^{-1}a)=0$; so we can order x_i's in such a way that

$$\left\|\sum_1^p \left(x_i'-\frac{1}{n}a\right)\right\|^2 \leqq \sum_1^n \left\|x_i-\frac{1}{n}a\right\|^2 \leqq \sum_1^n \left(\|x_i\|^2+2\frac{1}{n}\|x_i\|\,\|a\|+\frac{1}{n^2}\|a\|^2\right)$$

$$\leqq (2M+\|a\|)\|a\|+\sum \|x_i\|^2. \qquad \text{Q.E.D.}$$

We are now ready to prove Theorem 1. Let $X=\{x_n : n=1, 2, \ldots\}$ be the sequence of functions satisfying the conditions a–d. Each x_n can be written as $x_n=y_n+z_n$, $y_n \in M$, $z_n \in M^{\perp}=N$. It is clear that if $\{x_n'\}$ is a rearrangement of X, then $\sum x_n'$

converges in L_2 if and only if $\sum y'_n$ and $\sum z'_n$ converge in L_2. We shall show first that $\sum y'_n$ converges for any rearrangements of y's. For any $y \in M$ we have

$$\sum |(y_n, y)| = \sum |(x_n, y)| < \infty. \tag{13}$$

Hence $\sum (y_n, y)$ converges absolutely, so every rearrangement of $\sum (y_n, y)$ will converge to the same limit, say $W(y)$. This shows that $\sum y_n$ converges weakly, and for every $y \in M$, every subseries of $\sum (y_n, y)$ will also converge. This implies [1, p. 60] that every subseries of $\sum y_n$ will converge, which in turn implies [1, p. 59 (1-b)] that every rearrangement of $\sum y_n$ converges in norm. The strong limit of $\sum y_n$ must be equal to the weak limit of $\sum y_n$. The weak limit of $\sum y_n$ is the same for every rearrangement, hence the strong limit must be independent of the rearrangement. This shows that $\sum y_n$ converges unconditionally in norm, i.e. for every rearrangement it converges in norm to the same limit, say x_0. This proves the first part of the theorem, since if a rearrangement of $\sum x_n$ converges in norm then the limit must be of the form $x_0 + \sum z'_n$ where $\sum z'_n \in N = M^{\perp}$.

What remains to be shown is that for every $w \in N$ there exists a rearrangement of $\sum z_n$ which converges in norm to w. We introduce the following notation.

Let $W = \{w_i : i = 1, 2, \ldots\}$ be an arbitrary sequence of elements in N. Put

$P(W) = \{w_{i_1} + w_{i_2} + \cdots + w_{i_k} : i_1 < i_2 < \cdots < i_k\}$,

$Q(W) = \text{co}\, P(W)$ (convex hull of $P(W)$),

$R(W) = \{\gamma_1 w_{i_1} + \cdots + \gamma_k w_{i_k} : 0 \leqq \gamma_i \leqq 1,\ i_1 < i_2 < \cdots < i_k\}$.

Denote the elements of $P(W)$ by p. If $p = w_{i_1} + w_{i_2} + \cdots + w_{i_k}$, denote by $W - p$ the sequence $\{w_i : i \neq i_1, i_2, \ldots, i_k\}$. Put now W to be the sequence $\{z_i : i = 1, 2, \ldots\}$.

Let $z \in N$, $z \neq 0$. Then

$$\left|\sum (z, z_n)\right| = \left|\sum (z, x_n)\right| < \infty \quad \text{since } \sum x_n \text{ converges} \tag{14}$$

$$\sum |(z, z_n)| = \sum |(z, x_n)| = +\infty \quad \text{since } z \notin M. \tag{15}$$

Hence for any $z \neq 0$, $z \in N$, and any $T > 0$, there exist p_1, p_2 in $P(W)$ such that $(z, p_1) \geqq T$ and $(z, p_2) \leqq -T$. Lemma 4 shows then that $Q(W)$ is dense in N. Since $R(W) \supset Q(W)$ (Lemma 3), $R(W)$ is also dense in N. It follows then that the set $z + R(W) = \{z + r : r \in R(W)\}$ is also dense for every $z \in N$. The equations (14) and (15) also show that the sets $Q(W-p)$ and $z + R(W-p)$ are dense for every $p \in P(W)$ and $z \in N$. Hence we have shown

(A) for every $w \in N$, $z \in N$, $p \in P(W)$ and every $\varepsilon > 0$ there exists $r \in z + R(W-p)$ so that

$$\|w - r\| < \varepsilon. \tag{16}$$

Choose $w \in N$. We shall now construct a rearrangement of $\sum z_n$ which converges to w. Let

$$\varepsilon_k \downarrow 0, \qquad A_k^2 = \sum_k^{\infty} \|z_n\|^2 \quad \text{and} \quad B_k = \sup\{\|z_n\| : n \geqq k\}. \tag{17}$$

By the hypothesis of the theorem $A_k \downarrow 0$, $B_k \downarrow 0$. We shall define a sequence of elements $w_k \in N$ as follows. Let $p_1 = z_1$. Choose $q_1 \in p_1 + R(z - p_1)$ such that $q_1 = z_1 + u_1$, $u_1 \in R(W - p_1)$ and so that

$$\|w - p_1 - u_1\| < \varepsilon_1. \tag{18}$$

This is possible by (A). We have

$$u_1 = \gamma_1 z_{i_1} + \cdots + \gamma_j z_{i_j}, \qquad 0 \leqq \gamma_i \leqq 1, \qquad 1 < i_1 < \cdots < i_j.$$

Using Lemma 2 we can find $t_1 = \delta_1 z_{i_1} + \cdots + \delta_j z_{i_j}$, $\delta_i = 0$ or 1, such that

$$\|u_1 - t_1\|^2 \leqq \sum_1^j \|z_{i_n}\|^2 \leqq A_1^2. \tag{19}$$

Let $w_1 = z_1 + t_1 \in P(W)$. Clearly from (18), (19) we have

$$\|w - w_1\| \leqq \|w - z_1 - u_1\| + \|u_1 - t_1\| \leqq \varepsilon_1 + A_1. \tag{20}$$

We now put p_2 as the first z not used as a summand in w_1. Choose $q_2 \in p_2 + R(W - w_1 - p_2)$ such that $q_2 = p_2 + u_2$, $u_2 \in R(W - (p_1 + w_1))$ and

$$\|w - w_1 - p_2 - u_2\| \leqq \varepsilon_2.$$

Using Lemma 2 we choose $t_2 \in P(W - (p_2 + w_1))$ so that

$$\|u_2 - t_2\|^2 \leqq \sum_2^\infty \|z_i\|^2 = A_2^2.$$

Let $w_2 = p_2 + t_2 \in P(W - w_1)$. We get again

$$\|w - w_1 - w_2\| \leqq \|w - w_1 - p_2 - u_2\| + \|u_2 - t_2\| \leqq \varepsilon_2 + A_2.$$

Inductively it goes as follows. Suppose $w_1, w_2, \ldots, w_n$ are already defined and satisfy

$$\|w - (w_1 + \cdots + w_k)\| \leqq \varepsilon_k + A_k, \tag{21}$$

$$w_k \in P(W - (w_1 + \cdots + w_{k-1})), \qquad k = 1, 2, \ldots, n, \tag{22}$$

(23) $z_1, z_2, \ldots, z_k$ are included as summands in $w_1 + w_2 + \cdots + w_k$, $k = 1, 2, \ldots, n$.

Choose p_{n+1} to be the z with the smallest subscript not included in the sum $w_1 + \cdots + w_n$.

Let $q_{n+1} \in p_{n+1} + R(W - (p_{n+1} + w_1 + \cdots + w_n))$ be such that $q_{n+1} = p_{n+1} + u_{n+1} \in R(W - (p_{n+1} + \cdots + w_n))$ and

$$\|w - (w_1 + \cdots + w_n + p_{n+1} + u_{n+1})\| \leqq \varepsilon_{n+1}. \tag{24}$$

This is possible by (A). Using Lemma 2 choose $t_{n+1} \in P(W - (w_1 + \cdots + w_n + p_{n+1}))$ so that

$$\|u_{n+1} - t_{n+1}\|^2 \leqq \sum_{n+1}^\infty \|z_i\|^2 \leqq A_{n+1}^2. \tag{25}$$

Put $w_{n+1}=p_{n+1}+t_{n+1}\in P(W-(w_1+\cdots+w_n))$. We have from (24), (25)

$$\|w-(w_1+\cdots+w_{n+1})\| \leqq \varepsilon_{n+1}+A_{n+1}. \tag{26}$$

Clearly $w_1, w_2, \ldots, w_{n+1}$ satisfy (21), (22), (23). Thus the sequence w_n is defined, every z_i is included as a summand and in exactly one w_n and

$$\left\|w-\sum_1^n w_i\right\| \leqq \varepsilon_n+A_n. \tag{27}$$

This shows that $\sum w_n$ defines a rearrangement of $\sum z_n$, say $\sum z''_n$ such that

$$w_n = z''_{i_n}+\cdots+z''_{i_{n+1}-1} \tag{28}$$

and

$$\left\|w-\sum_1^{i_{n+1}-1} z''_i\right\| \leqq \varepsilon_n+A_n. \tag{29}$$

Now we rearrange each of the sums w_n in such a way that the resulting rearrangement $\sum z'_i$ will converge to w. Let

$$w_n = z''_{j_1}+\cdots+z''_{j_k}, \qquad n \leqq i_1 < i_2 < \cdots < i_k.$$

Using Lemma 5 we can rearrange z''_{j_1}s in such a way that

$$w_n = z'_{j_1}+\cdots+z'_{j_k} \tag{30}$$

$$\begin{aligned} \|z'_{j_1}+\cdots+s'_{j_p}\|^2 &\leqq \|z'_{j_1}\|^2+\cdots+\|z'_k\|^2+\|w_n\|(\|w_n\|+2B_n) \\ &\leqq A_n^2+\|w_n\|(\|w_n\|+2B_n), \qquad p = 1, 2, \ldots, k. \end{aligned} \tag{31}$$

Since

$$\begin{aligned} \|w_n\| &\leqq \|w-(w_1+\cdots+w_{n-1})\|+\|w-(w_1+\cdots+w_n)\| \\ &\leqq \varepsilon_{n-1}+A_{n-1}+\varepsilon_n+A_n = d_n \end{aligned}$$

we have

$$\|z'_{j_1}+\cdots+z'_{j_p}\|^2 \leqq A_n^2+d_n(d_n+B_n), \qquad p = 1, 2, \ldots, k. \tag{32}$$

Let now $\sum z'_i$ be the resulting rearrangement. Let N be a positive integer and choose $n=n(N)$ such that

$$w_{n+1} = z'_{i_1}+z'_{i_1+1}+\cdots+z'_{i_k}, \qquad i_1 \leqq N \leqq i_k.$$

Then

$$\begin{aligned} \left\|w-\sum_1^N z'_i\right\| &\leqq \left\|w-\sum_1^n w_i\right\|+\|z'_{i_1}+\cdots+z'_N\| \\ &\leqq \varepsilon_n+A_n+[A_n^2+d_n(d_n+B_n)]^{1/2}. \end{aligned} \tag{33}$$

It is clear that $n(N)\to\infty$ as $N\to\infty$; so the right hand side of (33) tends to 0 as $N\to\infty$. This completes the proof of Theorem 1.

3. Examples.

1. $M=\{y : \sum |(x_n, y)| < \infty\}$ need not be closed, even if the conditions a–c of Theorem 1 hold.

Let $\{e_n : n=1, 2, \dots\}$ be an orthonormal basis of L_2. Put

$$l_n = [(n+2)\log(n+2)]^{-1} \qquad x_n = (-1)^{n+1}l_n(e_1+e_2+\cdots+e_n)$$

$$x = \sum_{i=1}^{\infty}\sum_{j=i}^{\infty}(-1)^{j+1}l_j e_i = \sum_{i=1}^{\infty}\alpha_i e_i. \tag{34}$$

It follows that

$$\|x_n\|^2 = nl_n^2 < [n(\log n)^2]^{-1} \tag{35}$$

so $\sum \|x_n\|^2 < \infty$, $\sum \|x_n\| = +\infty$. Since $\sum(-1)^{n+1}l_n$ is an alternating series, we have the following estimate

$$\left|\sum_{i=p}^{\infty}(-1)^{n+1}l_i - \sum_{i=p}^{p+q}(-1)^{n+1}l_i\right| < |l_{q+1}|. \tag{36}$$

Hence

$$\sum_{i=1}^{\infty}\left|\sum_{j=1}^{\infty}(-1)^{n+1}l_n\right|^2 = \sum_{i=1}^{\infty}\left|\sum_{j=1}^{\infty}(-1)^{n+1}l_n - \sum_{j=1}^{i=1}l_n\right|^2 \leqq \sum_{i=1}^{\infty}|l_i|^2 < \infty \tag{37}$$

so $x \in L_2$. Moreover using (36) we obtain

$$\left\|\sum_1^N x_n - x\right\|^2 = \left\|\sum_1^N x_n - \sum_1^N \alpha_n e_n\right\|^2 + \left\|\sum_{N+1}^{\infty}\alpha_n e_n\right\|^2. \tag{38}$$

The last term clearly goes to 0 as $N \to \infty$. The first term is equal to

$$\sum_{n=1}^{N}\left|\sum_{i=n}^{N}(-1)^{n+1}l_i - \sum_{i=n}^{\infty}(-1)^{n+1}l_n\right|^2 \leq \sum_{n=1}^{N}|\,l_{N-1}\,| = Nl_{N+1} \to 0 \qquad (N \to \infty).$$

Hence $\sum x_n$ converges to x in norm. Put now

$$t_i = [\log(\log(i+2))]^{-1}$$

$$y_1 = 0, \qquad y_k = t_1e_1 + \sum_{i=1}^{k-1}(t_{i+1}-t_i)e_i - t_ke_k, \qquad k \geqq 2, \tag{39}$$

$$y = t_1e_1 + \sum_{i=1}^{\infty}(t_{i+1}-t_i)e_i.$$

Since

$$t_{i+1}-t_i = \int_i^{i+1}\frac{d}{dx}\frac{1}{\log(\log(x+2))}\,dx$$

$$= -\int_i^{i+1}[(x+2)\log(x+2)(\log(\log(x+2)))^2]^{-1}\,dx \tag{40}$$

so

$$|t_{i+1}-t_i| \leqq \frac{1}{(i+2)\log(i+2)[\log(\log(i+2))]^2}$$

and hence $y \in L_2$. Moreover

$$\|y-y_k\|^2 \leqq |t_k|^2+|t_{k+1}-t_k|^2+\sum_{i=k}^{\infty}|t_{i+1}-t_i|^2 \to 0 \qquad (k\to\infty)$$

so y_k converges in norm to y. We also note that $|(x_n, y_k)|=0$ for $n>k$, hence $\sum |(x_n, y_k)|<\infty$ for all k. However $|(x_n, y)|=l_n t_n$ and since

$$\sum \frac{1}{(i+2)\log(i+2)\log(\log(i+2))} = \infty \qquad y \notin M,$$

this shows that M need not be closed.

2. Next we give an example of a series $\sum x_n$ for which $N=L_2$, or equivalently $M=\{0\}$. Let $\{e_n\}$ be an orthonormal basis of L_2. Define

$$x_n^{(k)} = (-1)^{n+1}2^{-k}n^{-1}e_k. \tag{41}$$

Since $\|x_n^{(k)}\|^2=2^{-2k}n^{-2}$ then $\sum_n\sum_k \|x_n^{(k)}\|^2<\infty$. It is also clear that $\sum_{n,k}\|x_n^{(k)}\|=\infty$. We can order $\{x_n^{(k)}\}$ into a single sequence $\{x_n\}$ such that $\sum x_n$ converges. We proceed by induction as follows. Let $x_1=x_1^{(1)}$, $x_2=x_1^{(2)}$ and $x_3=x_2^{(1)}$. Suppose we have ordered all the elements $x_n^{(k)}$, $n+k\leqq m$, into a sequence $x_1, x_2, \ldots, x_N$ such that if x_i corresponds to $x_n^{(k)}$, x_j corresponds to $x_n^{(k+1)}$ then $i<j$. We put

$$x_{N+1} = x_1^{(m)}, \qquad x_{n+2}^{(m)} = x_2^{(m-1)}, \ldots, x_{N+j} = x_j^{(m-j+1)}, \qquad x_{N+m} = x_m^{(1)}. \tag{42}$$

In this way we have ordered all the elements $\{x_n^{(k)}\}$ in such a way that if x_i corresponds to $x_n^{(k)}$ and x_j corresponds to $x_n^{(k+1)}$ then $i<j$. Hence for each k

$$\sum_i (x_i, e_k) = \frac{1}{2^k}\sum_j \frac{(-1)^{j+1}}{j} = \frac{1}{2^k}\log 2 = y_k.$$

Put

$$x = \sum_k \frac{1}{2^k}\log 2e_k = \sum_k y_k e_k. \tag{43}$$

We have for any positive integer N

$$\left|\sum_{i=1}^{N}(-1)^{i+1}\frac{1}{2^k i}-\frac{1}{2^k}\log 2\right| \leqq \frac{1}{2^k N}. \tag{44}$$

Let N be a positive integer and choose p to be the largest integer such that $x_1^{(p)}$ is included among $x_1, x_2, \ldots, x_N$. Then by the construction of x_i's

$$\left\|\sum_{i=1}^{N} x_i - x\right\| \leqq \left\|\sum_1^p x_i-\sum_1^p y_i e_i\right\|+\left\|\sum_{p+1}^{N} x_i-\sum_{p+1}^{N} y_i e_i\right\|+\left\|\sum_{N+1}^{\infty} y_i e_i\right\|. \tag{45}$$

The last sum clearly goes to 0 as $N \to \infty$. By (44) the first and the second sum are each smaller than

$$\frac{1}{p}\frac{1}{1}+\frac{1}{p-1}\frac{1}{2}+\frac{1}{p-2}\frac{1}{2^2}+\cdots+\frac{1}{1}\frac{1}{2^{p-1}} = S_p$$

which goes to 0 as $p \to \infty$. Since $p \to \infty$ as $N \to \infty$ we have shown that $\sum x_i$ converges to x in norm. Let now $y \neq 0$, $y \in L_2$, $y = \sum b_k e_k$. Then

$$\sum |(x_i, y)| \geqq \sum_n |(x_n^{(k)}, y)| = \sum_n \frac{b_k}{2^k n} = \infty \tag{46}$$

unless $b_k = 0$. Hence $M = \{0\}$ and every function in L_2 can be obtained as a limit of some rearrangement of $\sum x_i$.

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